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A Note on a Theorem of A. Granville and K. Ono

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It was recently proved by A. Granville and K. Ono that if $t \in \mathbb{N}$, $t \geqslant 4$ then every natural number has a t-core partition. The essence of the proof consists in showing this assertion for t prime, $t \geqslant 11$. We give an alternative, short proof for these

Suppose that $n \in \mathbb{N}$ and that $\lambda = (\lambda_1, \lambda_2, ...)$ is a partition of n, i.e., $\lambda_1, \lambda_2, ...$ is a sequence of non-negative integers of which only finitely many are $\neq 0$, and such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \qquad \text{and} \qquad n = \sum_{i=1}^{\infty} \lambda_i.$$

Then the partition $\lambda' = (\lambda'_1, \lambda'_2, ...)$ associated with λ is defined by

$$\lambda'_i = \#\{ j \in \mathbb{N} \mid \lambda_j \geqslant i \},$$

and λ' is also a partition of n. If $(i, j) \in \mathbb{N}^2$ such that $\lambda_i \lambda'_j > 0$ we have the (i, j)th hook number of λ:

$$h_{i,j}(\lambda) := \lambda_i + \lambda'_j - i - j + 1.$$

If $t \in \mathbb{N}$ and if none of the hook numbers of λ is divisible by t, one says that λ is a t-core partition of n.

The "t-core partition conjecture" asserts that if $t \in \mathbb{N}$, $t \geqslant 4$, then every $n \in \mathbb{N}$ has a t-core partition. This conjecture is proved by A. Granville and K. Ono in the papers [2], [3]. The conjecture had attracted some interest since it has implications in the representation theory of symmetric and alternating groups: It implies that if p is a prime number $\geqslant 5$ then for every $n \in \mathbb{N}$ both the symmetric group S_n and the alternating group A_n has a

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p-block of defect zero. Together with previously known results this further implies that if p is prime $\geqslant 5$ then every finite simple group has a p-block of defect zero. We refer to [2] for a survey of these results.

Now let $n \in \mathbb{N}$. Then n has a t-core partition if and only if the equations

$$n = \sum_{i=0}^{t-1} \left(\frac{t}{2} \cdot x_i^2 + i x_i \right) \qquad \text{and} \qquad \sum_{i=0}^{t-1} x_i = 0 \tag{0}$$

have a (simultaneous) integral solution; in fact, it was proved in [1] that the number of integral solutions to (0) equals the number of t-core partitions of n.

As explained in [3], the t-core partition conjecture is proved if it is shown that (0) has for all $n \in \mathbb{N}$ an integral solution in each of the cases: $t = 4, 5, 6, 7, 9$ and (t prime, $t \geqslant 11$). The cases $t = 4, 6, 9$ were handled in [3] by special arguments. The cases $t = 5, 7, 11, 13$ can be handled by using modular forms: K. Ono informed me that the cases $t = 5$, 7 and $t = 11$ were done in unpublished notes by A. O. L. Atkin and himself respectively. Finally, in [2] a special argument using modular forms for the case $t = 13$ and a general argument for $t \geqslant 17$ were given.

We shall give an alternative, simple proof of the fact that (0) has an integral solution for all $n \in \mathbb{N}$ if t is prime, $t \geqslant 11$; in particular, our proof avoids computation with modular forms. As in [2], the proof has two parts: One for "large" n and one for "small" n. We pay for the simplicity of our argument for "large" n by having to prove the following lemma, which requires a small amount of computation (which however is not more than the amount of computation which was needed for the case $t = 13$ in [2]).

LEMMA. *Suppose that t is prime, $t \geqslant 11$ and $n \leqslant t(t^2-1)/4 + (t-1)$. Then* (0) *has an integral solution* $(x_0, ..., x_{t-1})$.

Proof. Put $n_0 = t(t^2-1)/4 + (t-1)$. Suppose that $s \in \{1, ..., (t-1)/2\}$; put

$$m_s := \frac{t+1}{2} - s, \qquad c_0(n) := n;$$

define the integers $y_i(n)$, $c_i(n)$ for $i = 1, ..., s$ successively,

$$y_i(n) := [(-1 + \sqrt{1 + 4tc_{i-1}(n)})/2t], \quad c_i(n) := c_{i-1}(n) - (ty_i(n)^2 + y_i(n));$$

so that $c_i(n) \geqslant 0$ for all i; and define the integers $x_{m_s}(n), ..., x_{t-m_s}(n)$:

$$x_{m_s+2i-2}(n) := -y_i(n), \qquad x_{m_s+2i-1}(n) := y_i(n) \qquad \text{for} \quad i = 1, ..., s.$$

Then we have

$$c_s(n) = n - \sum_{i=m_s}^{t-m_s} \left(\frac{t}{2} \cdot x_i(n)^2 + i x_i(n) \right) \qquad \text{and} \qquad \sum_{i=m_s}^{t-m_s} x_i(n) = 0.$$

We conclude that the proof is finished if for some $s \in \{1, ..., (t-1)/2\}$ the equations

$$r = \left(\sum_{i=0}^{m_s - 1} + \sum_{i=t-m_s+1}^{t-1} \right) \left(\frac{t}{2} \cdot x_i^2 + i x_i \right) \qquad \text{and} \qquad \left(\sum_{i=0}^{m_s - 1} + \sum_{i=t-m_s+1}^{t-1} \right) x_i = 0 \tag{1}$$

have an integral solution for $r = c_s(n)$ for all $n \leqslant n_0$.

Now let us notice that the proof of Lemma 1 in [2] actually shows that if $r \leqslant m_s(m_s - 1)$ then (1) has an integral solution (with $x_i \in \{0, \pm 1\}$ for all i).

So the proof is finished if for some $s \in \{1, ..., (t-1)/2\}$ we have

$$c_s(n) \leqslant \left(\frac{t+1}{2} - s \right) \left(\frac{t-1}{2} - s \right) \qquad \text{for all} \quad n \leqslant n_0. \tag{2}$$

Now, $y_i(n) \in \mathbb{Z}$ is the largest possible such that $t y_i(n)^2 + y_i(n) \leqslant c_{i-1}(n)$, and so

$$\begin{aligned} c_i(n) &= c_{i-1}(n) - (t y_i(n)^2 + y_i(n)) \leqslant 2t y_i(n) + t + 1 \\ &\leqslant t + \sqrt{1 + 4t c_{i-1}(n)}; \end{aligned}$$

consequently, if $s \in \{1, ..., (t-1)/2\}$, $j \in \{0, ..., s-1\}$ and $f_j(x)$ is a polynomial, then the condition

$$(4t)^{2^j - 1} c_{s-j}(n) \leqslant f_j(t) \qquad \text{for all} \quad n \leqslant n_0$$

is implied by

$$(4t)^{2^{j+1} - 1} c_{s-j-1}(n) \leqslant f_{j+1}(t) \qquad \text{for all} \quad n \leqslant n_0,$$

where

$$f_{j+1}(x) := (f_j(x) - x \cdot (4x)^{2^j - 1})^2 - (4x)^{2^{j+1} - 2}, \tag{3}$$

provided that

$$g_j(t) := f_j(t) - t \cdot (4t)^{2^j - 1} \geqslant 0. \tag{4}$$

Hence, if we define $f_0(x) := ((x+1)/2-4)((x-1)/2-4)$, $g_0(x) := f_0(x)-x$, and $f_i(x)$, $g_i(x)$ for $i=1, 2, 3, 4$ in accordance with (3) and (4), we have that (2) holds for the case $s=4$, if

$$g_0(t),\ g_1(t),\ g_2(t),\ g_3(t) \geqslant 0 \text{ and } ((4t)^{15} c_0(n) \leqslant f_4(t) \text{ for all } n \leqslant n_0). \tag{5}$$

But since we have $c_0(n) = n \leqslant t(t^2-1)/4 + (t-1)$, we see that (5) holds if

$$g_0(t), \ldots, g_3(t),\ f(t) \geqslant 0, \tag{6}$$

where

$$f(x) := f_4(x) - (4x)^{15}\,(x(x^2-1)/4 + x - 1).$$

Now, $g_0(x), \ldots, g_3(x)$, $f(x)$ are certain polynomials with rational coefficients which can easily be computed by using (for example) MAPLE. For example, one finds

$$f(x) = 2^{-32}(x^{32} - 320x^{31} + 48496x^{30} - 4639040x^{29} + \cdots).$$

Again, using MAPLE, we can compute approximations to the real roots of these polynomials and thus verify that (6) holds for $t \geqslant 43$, i.e., (2) holds for $s=4$ if $t \geqslant 43$.

On the other hand, when $t \in \{37, 41\}$, (2) holds in the case $s=4$ since a direct computation shows that for all $n \leqslant n_0$ we have

$$(c_4(n) \leqslant 123 \text{ if } t=41), \qquad (c_4(n) \leqslant 111 \text{ if } t=37);$$

hence we may assume that $t \leqslant 31$.

In the remaining cases, $t = 11, 13, 17, 19, 23, 29, 31$, we finish the proof by stating the following facts which can be checked in a few minutes on a machine.

If $t \in \{11, 13\}$ and $n \leqslant n_0$, (0) has a solution with

$$x_1, x_2, x_3, x_4, x_6 \in \{0, \pm 1\} \qquad \text{and} \qquad x_{t-4}, x_{t-3}, x_{t-2}, x_{t-1} \in \{0, \pm 1, \pm 2\} \tag{7}$$

and

$$x_5 = 0 \qquad \text{and} \qquad x_j = 0 \qquad \text{for} \quad 7 \leqslant j \leqslant t-5. \tag{8}$$

For $t = 17, 19, 23, 29, 31$ we have $c_1(n) \leqslant c_1 := 255, 323, 483, 783, 899$ respectively, and (1) has for $s=1$ and all $r \leqslant c_1$ a solution with (7) and (8). Q.E.D.

We can now give our alternative proof of the following theorem due to Granville and Ono, cf. [2, 3]. In their proof of existence of integral solutions to (0) when $t \geqslant 17$, they exploited the theorem of Lagrange on the representation of natural numbers as sums of four squares. In our proof of existence for $t \geqslant 11$ (t prime) we use Gauss' three square theorem.

THEOREM. *Suppose that t is prime, $t \geqslant 11$.*

Then for all $n \in \mathbb{N}$, (0) has an integral solution.

Proof. Write $n = tm' + r'$ with $0 \leqslant r' \leqslant t-1$. Because of the lemma we may and will assume $n \geqslant t(t^2-1)/4 + (t-1)$, hence $m' \geqslant (t^2-1)/4$. Define the integers m and r as follows:

$$(m, r) := \begin{cases} (m', r') & \text{if} \quad m' \equiv 1\ (2),\ r' \not\equiv 0\ (4) \\ (m'+2, r'-2t) & \text{if} \quad m' \equiv 1\ (2),\ r' \equiv 0\ (4) \\ (m' \mp 1, r' \pm t) & \text{if} \quad m' \equiv 0\ (2),\ r' \equiv \pm t\ (4) \\ (m'+1, r'-t) & \text{if} \quad m' \equiv r' \equiv 0\ (2). \end{cases}$$

Then $n = tm + r$ and either

$$m \equiv r \equiv 1\ (2) \qquad \text{and} \qquad 4m \geqslant r^2, \tag{I}$$

or

$$m \equiv 1\ (2), \qquad r \equiv 2\ (4) \qquad \text{and} \qquad 16m \geqslant r^2. \tag{II}$$

If case (I) prevails, we have that $4m - r^2$ is odd and $\not\equiv -1\ (8)$. Hence, by Gauss' three square theorem we have

$$4m - r^2 = a^2 + b^2 + c^2$$

for certain integers a, b, c which must all be odd (since $4m - r^2 \equiv 3\ (4)$). We may then assume that $r + a + b + c$ is divisible by 4, and we define the integers

$$\alpha = (r+a+b+c)/4, \qquad \beta = (r-a-b+c)/4,$$

$$\gamma = (r-a+b-c)/4, \qquad \delta = (r+a-b-c)/4,$$

$$x_0 = -\alpha,\ x_1 = \alpha,\ x_2 = -\beta,\ x_3 = \beta,\ x_4 = -\gamma,\ x_5 = \gamma,\ x_6 = -\delta,\ x_7 = \delta,$$

and $x_i = 0$ for $i \geqslant 8$. Then $x_0 + \cdots + x_{t-1} = 0$ and:

$$\sum_{i=0}^{t-1} \left(\frac{t}{2} \cdot x_i^2 + i x_i \right) = t \cdot \frac{1}{4}(r^2 + a^2 + b^2 + c^2) + r = tm + r = n.$$

In case (II), use the same arguments and definitions of α, β, γ, δ with r replaced by $r/2$. Then put

$$x_0 = -\alpha,\ x_1 = -\beta,\ x_2 = \alpha,\ x_3 = \beta,\ x_4 = -\gamma,\ x_5 = -\delta,\ x_6 = \gamma,\ x_7 = \delta,$$

and $x_i = 0$ for $i \geqslant 8$. Q.E.D.

Remark. The above lemma could of course be checked for all odd t in the range $9 \leqslant t \leqslant 41$. This would prove the lemma and the above theorem for odd t with $t \geqslant 9$.

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